# Control Engineering 2015-2016
# Exam 3 November 2015
## Prof. C. De Persis

- You have **3 hours** to complete the exam.

- You **can** use books and notes but **not** smart phones, computers, tablets and the like.

- For each problem, please do your best to write your answers within the box that is below the problem text. Notice that the box extends over the page following the one where the problem is stated.

- There are questions/exercises labeled as **Bonus**. These questions/exercises are optional and give you **extra** points if answered correctly.

- Please write down your Surname, Name, Student ID on each sheet.

- You will be given 2 sheets. If you need more, please ask. Please hand in **all the sheets** that you have used and the **text of the exam**.

- If you return the sheets, then your exam will be graded, unless you explicitly write "do not grade" on the first page.

### For the grader only

|  | Exercise 1 | Exercise 2 | Exercise 3 | Exercise 4 | Exercise 5 |
|---|---|---|---|---|---|
| **Points** |  |  |  |  |  |
| **Bonus** | $\star\star\star\star$ |  |  | $\star\star\star\star$ |  |

1. [15pts] The goal of this problem is to derive the equations of motion for the cart-pendulum system depicted in Figure 1 using the Euler-Lagrange equations of motion. The generalized coordinate of the system is $q = (p, \theta)$ and the generalized velocity is $\dot{q} = (\dot{p}, \dot{\theta})$.

   To the end of obtaining these equations of motion, answer the following questions:

   (a) [2pts] Write the horizontal position $x_1$ and the vertical position $x_2$ of the mass $m$ in terms of $p, l, \theta$ and use it to obtain the position vector $x \in \mathbb{R}^2$ of the mass $m$.

   (b) [2pts] Take the derivative of both components of $x$ to obtain the velocity vector $\dot{x} \in \mathbb{R}^2$.

   (c) [2pts] Write down the total kinetic co-energy $T_m^*(q, \dot{q}) = \frac{1}{2} m \dot{x}^\top \dot{x}$ of the mass $m$.

   (d) [2pts] Write down the total kinetic co-energy of the system by summing up the translational kinetic co-energy $T_c^*(\dot{q})$ of the cart, the total kinetic co-energy $T_m^*(\dot{q})$ of the mass $m$ and the rotational kinetic co-energy of the pendulum. To write the latter term, denote the moment of inertia of the pendulum by $J$.

   **Hint** For this and the remaining questions of the problem, if you did not answer question (c), then use the expression

   $$T_m^*(q, \dot{q}) = \frac{1}{2}(a\dot{p}^2 + b\dot{\theta}^2) + c\cos\theta\,\dot{\theta}\,\dot{p},$$

   where $a, b, c$ are suitable constants.

   (e) [1 pt] Determine the vector $\tau$ of external generalized forces.

   (f) [1 pt] Bearing in mind that the potential energy of the system stored in the mass $m$ and due to the work of the gravity force is

   $$V(q) = mgl(\cos\theta + 1),$$

   determine the Lagrangian function $L(q, \dot{q})$.

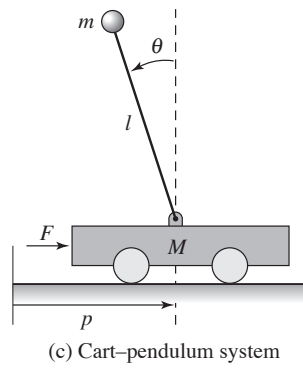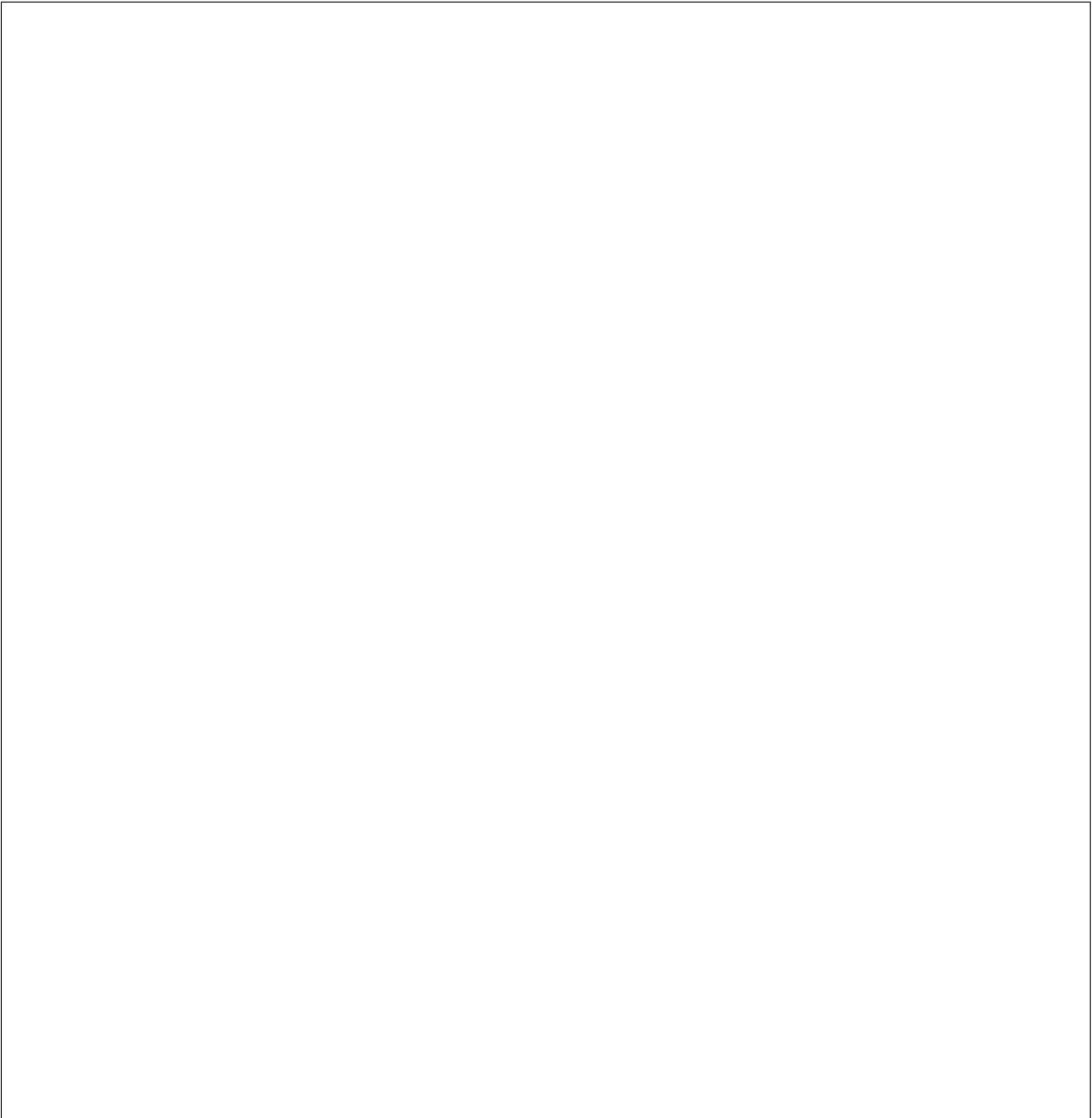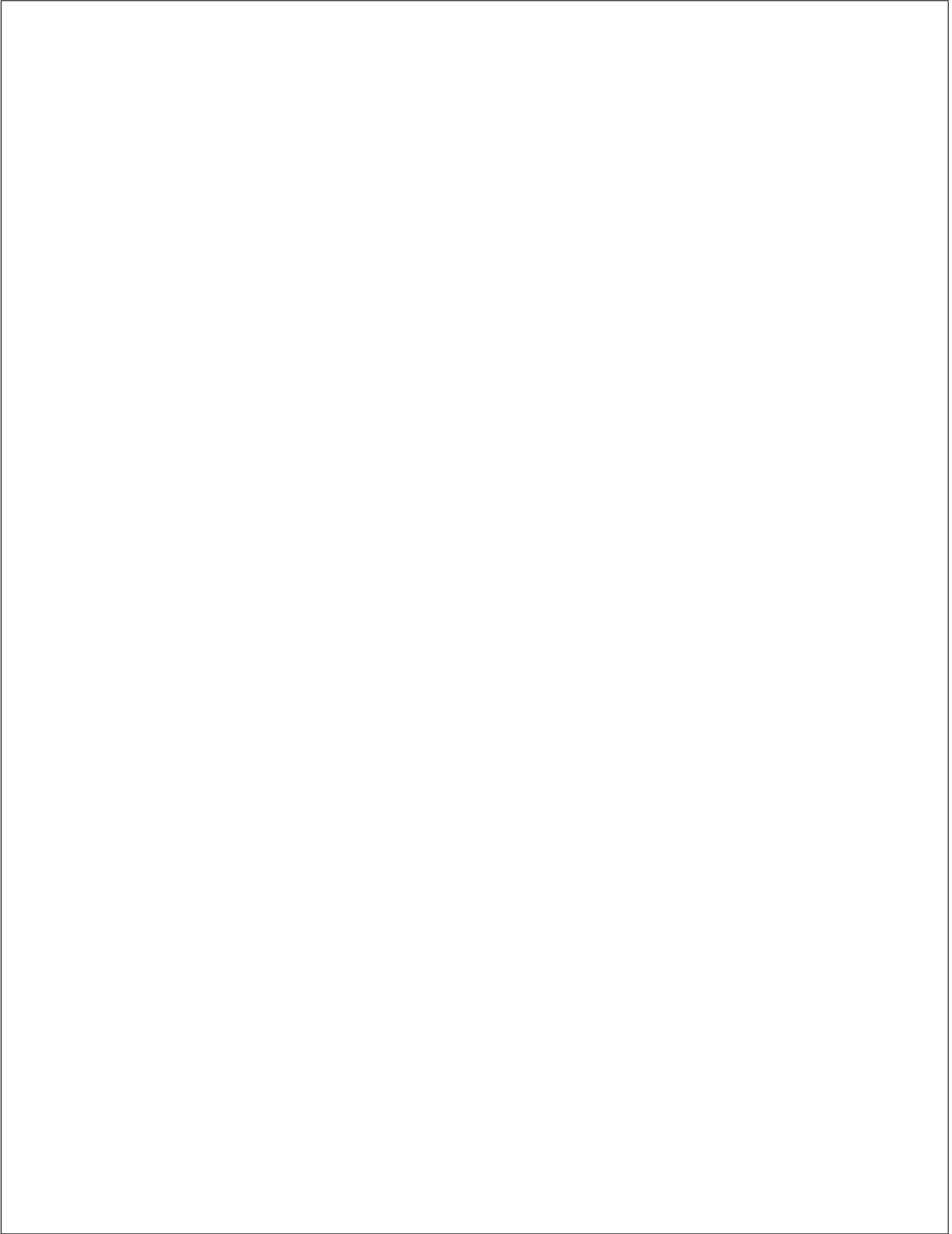   (g) [5 pts] Use the Euler-Lagrange equations of motion, to determine a dynamic model of the pendulum-cart system.

(c) Cart–pendulum system

Figure 1: From Aström-Murray, p. 25.

2. [15pts] [Textbook, Exercise 3.10, revised] (**Fisheries management**)[1] The dynamics of commercial fishery can be described by the following equations

$$\begin{aligned}
\dot{x} &= rx(1 - \frac{x}{k}) - axu := f(x, u) \\
y &= baxu - cu := h(x, u)
\end{aligned} \tag{1}$$

where:

- $x \in \mathbb{R}$ is the total biomass;
- $u \in \mathbb{R}$ is the control input;
- $y \in \mathbb{R}$ is the rate of revenue;
- $a, b, c, k, r$ are positive constants;
- $rx(1 - \frac{x}{k})$ is the biomass growth rate;
- $axu$ is the harvesting rate.

Answer the following questions:

(a) [3.5pts] Given a constant input $u_e$, determine all the equilibria $x_e$ of the system. For the non-zero equilibrium, calculate the corresponding rate of revenue $y_e$.

(b) [2.5pts] Determine the value of $u_e$ that gives the maximum rate of revenue. Then determine the corresponding equilibrium state $x_e$.
   **Hint** To answer this question remember that the maximum of a parabola $\alpha z^2 + \beta z + \gamma$ with $\alpha < 0$ is achieved at $z_{\max} = -\frac{\beta}{2\alpha}$ and is equal to $-\frac{\beta^2}{4\alpha} + \gamma$.

(c) [3pts] Design the constant input $u_e$ such that the resulting non-zero equilibrium as determined in Question (a) is given by $x_e = \frac{c}{ab}$.

(d) [3pts] Linearize the dynamics of the fishery around the equilibrium pair[2] $(x_e, u_e)$. Then specialize the answer to the case in which $(x_e, u_e)$ are those determined in Question (b). If you did not answer Question (b), then set $(x_e, u_e) = (\frac{c}{ab}, \frac{cr}{a^2 b})$.

(e) [3pts] For the linearized system obtained in Question (d), namely

$$\begin{aligned}
\dot{\delta x} &= A\delta x + B\delta u \\
\delta y &= C\delta x + D\delta u
\end{aligned}$$

set $\delta u = 0$ (this corresponds to set $u = u_e$). Determine whether the origin of the resulting system is asymptotically stable, stable or unstable. Do the solution of the original nonlinear system (1) that starts sufficiently close to the equilibrium $x_e$ converge to it? Explain.

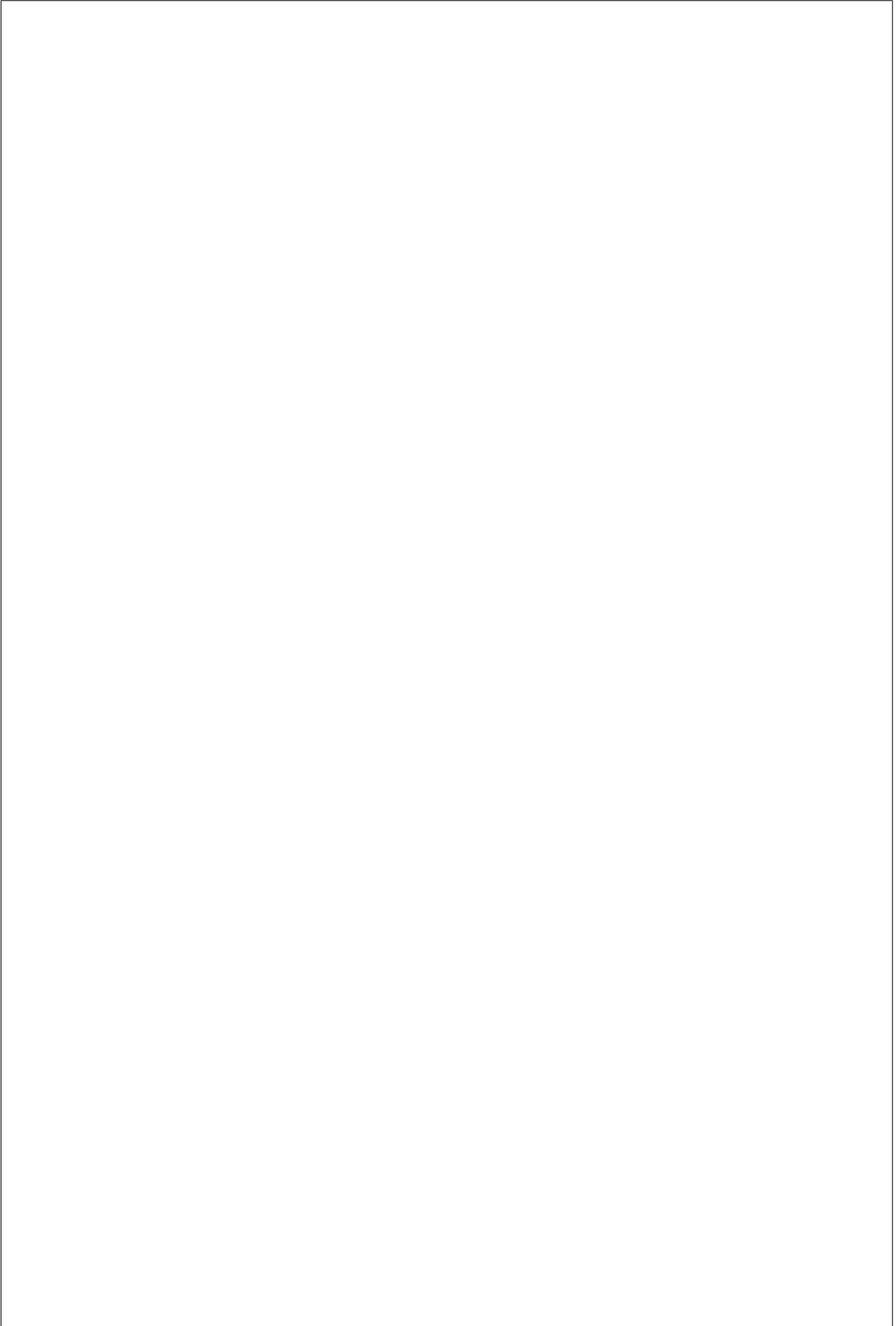(f) [3pts] (**Bonus**) For the linearized system obtained in Question (d), namely

$$\begin{aligned}
\dot{\delta x} &= A\delta x + B\delta u \\
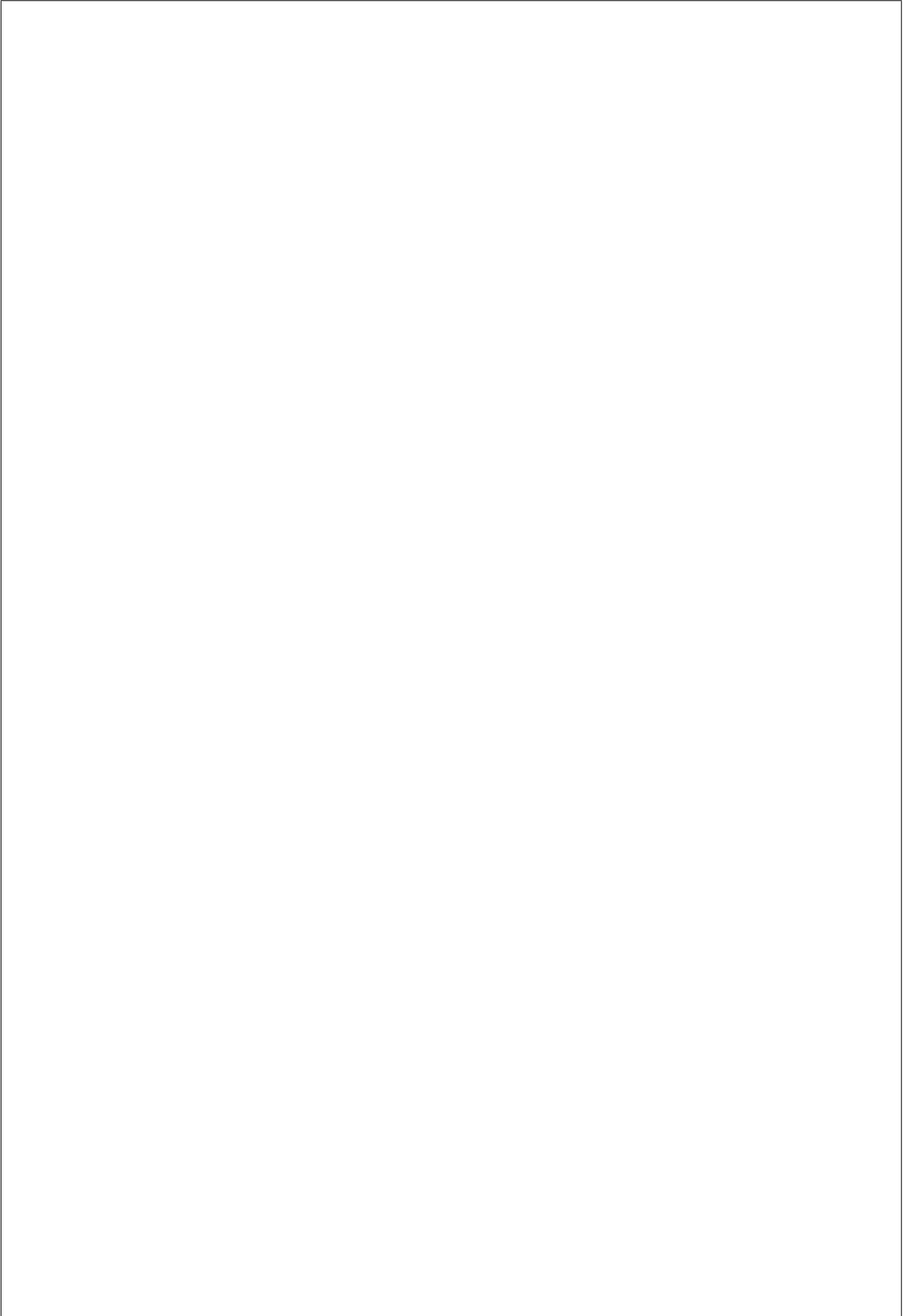\delta y &= C\delta x + D\delta u
\end{aligned}$$

compute the output response from an initial condition $\delta x(0) = 1$ and under a step input $\delta u(t) = 1$.

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[1]The textbook is not necessary for the resolution of this problem.
[2]Here $(x_e, u_e)$ are constants, without a specific numeric value, thus you can answer this question even if you did not solve the previous questions.
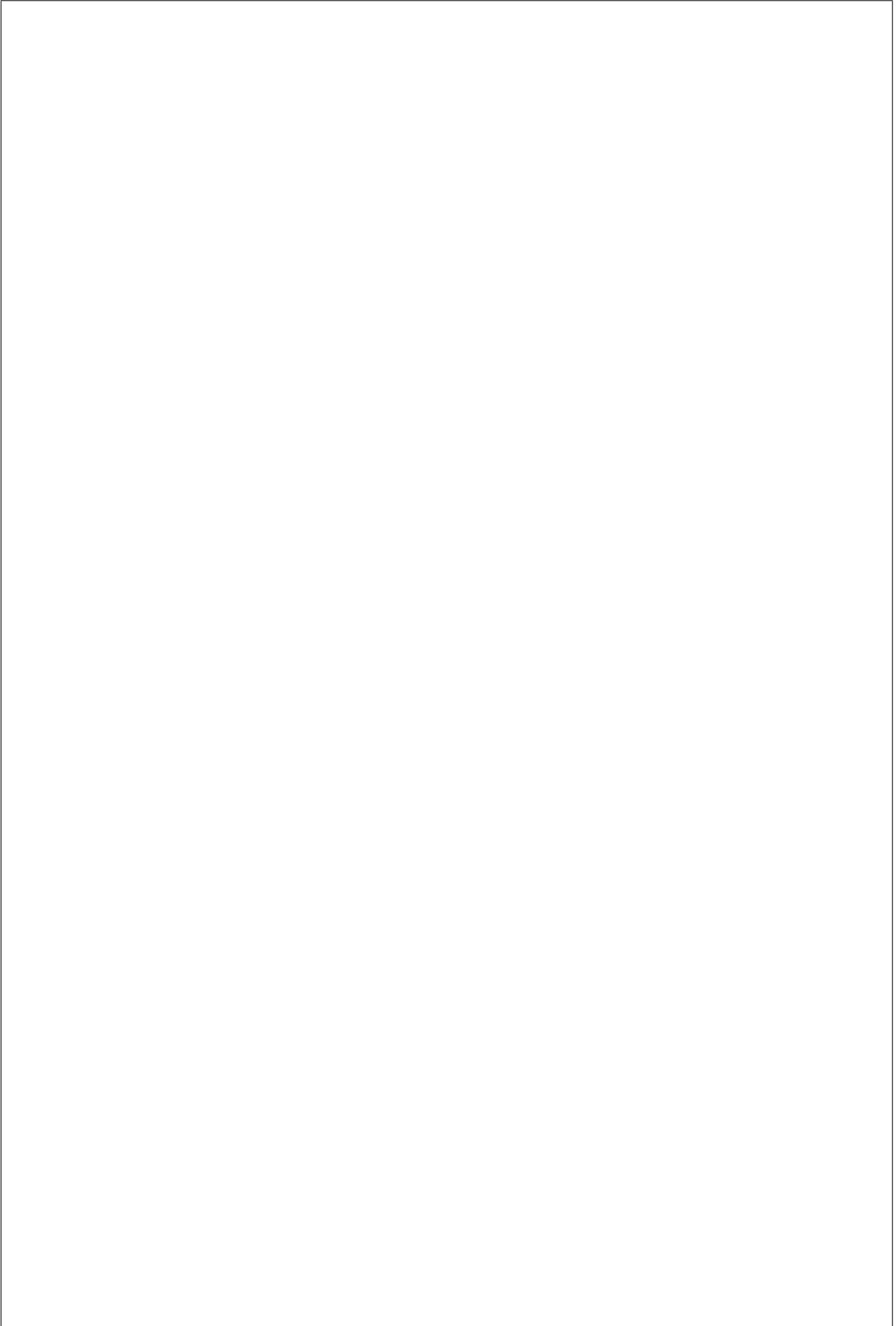
3. [10pts] Consider the normalized turbine-governor dynamics of a synchronous generator given by
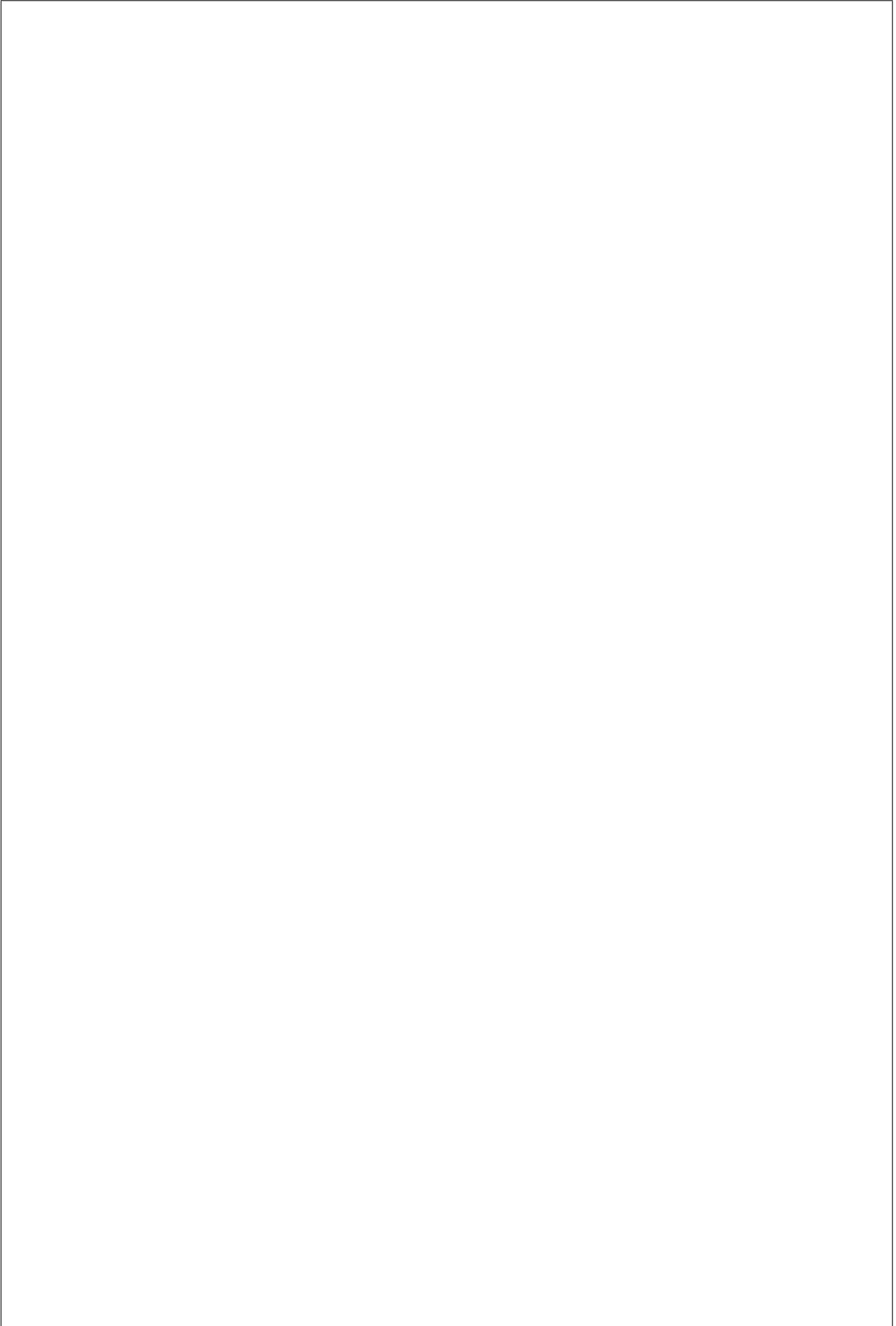
$$\dot{x} = Ax + Bu = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

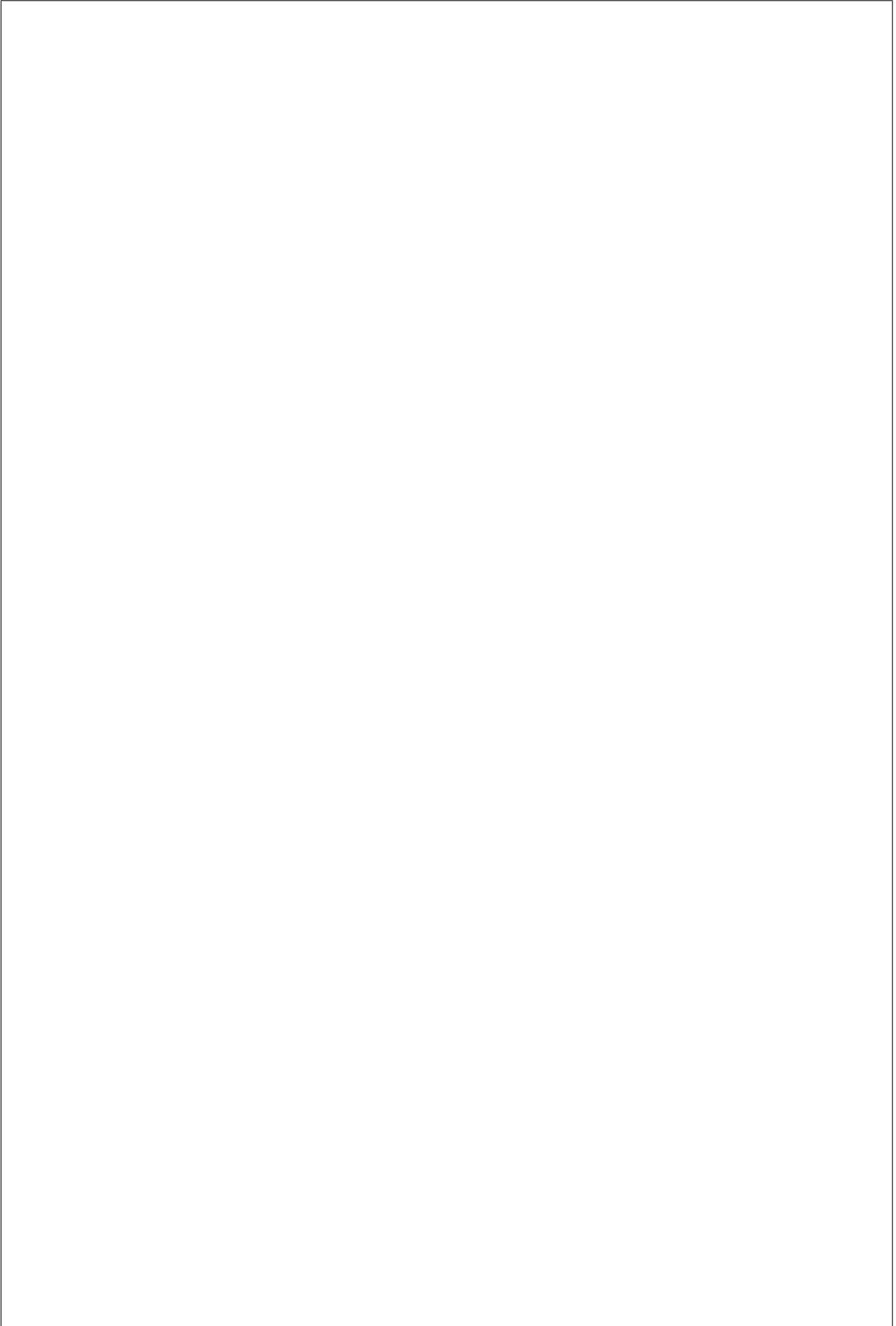$$y = Cx = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$ 

(2)

where $x_1 \in \mathbb{R}$ is the mechanical power produced by the generator, $x_2 \in \mathbb{R}$ is another physical variable, and $u \in \mathbb{R}$ is the control input.

(a) [1pt] Determine the reachability matrix $W_r$ and discuss whether the system is reachable or not.

(b) [1pt] Determine the reachable canonical form of the state space equation.

(c) [1pt] Determine the reachability matrix $\tilde{W}_r$ of the reachable canonical form.

(d) [1pt] Using Table 6.1 of your textbook, determine the eigenvalues for which the output step response of the system (2) in closed-loop system with the state feedback $u = -Kx + k_r r$ has an overshoot $M_p = 4\%$ and a 2% settling time of 11.8 units of time.

(e) [2pts] Determine the gain matrix $K$ such that the eigenvalues of $A - BK$ are equal to the eigenvalues determined in Question (d). Write explicitly the feedback $u = -Kx + k_r r$ which guarantees the closed-loop system to have the eigenvalues at the desired location and its output response to converge asymptotically to $r$.
**Hint** If you did not answer Question (d), then assign the eigenvalues $\{-2, -2\}$.

(f) [1pt] Determine the observability matrix $W_o$ of system (2) and discuss whether the system is observable or not.

(g) [1pt] Write the observable canonical form and compute the observability matrix $\tilde{W}_o$ of the system in observable canonical form.

(h) [2pts] Determine the observer gain $L$ designed according to Theorem 7.2. Namely, determine the gain $L$ that makes the characteristic polynomial of $A - LC$ to coincide with the polynomial $(s + 5)^2$.

(i) [3pts] **(Bonus)** Give the explicit expression of the matrices $F, G, H, K, J$ of the dynamical controller

$$\dot{\hat{x}} = F\hat{x} + Gy + Hr$$
$$u = K\hat{x} + Jr$$

that assigns the characteristic polynomial $\det(sI - A + BK) \cdot \det(sI - A + LC)$ and that guarantees the output response of the closed-loop system to converge asymptotically to $r$.
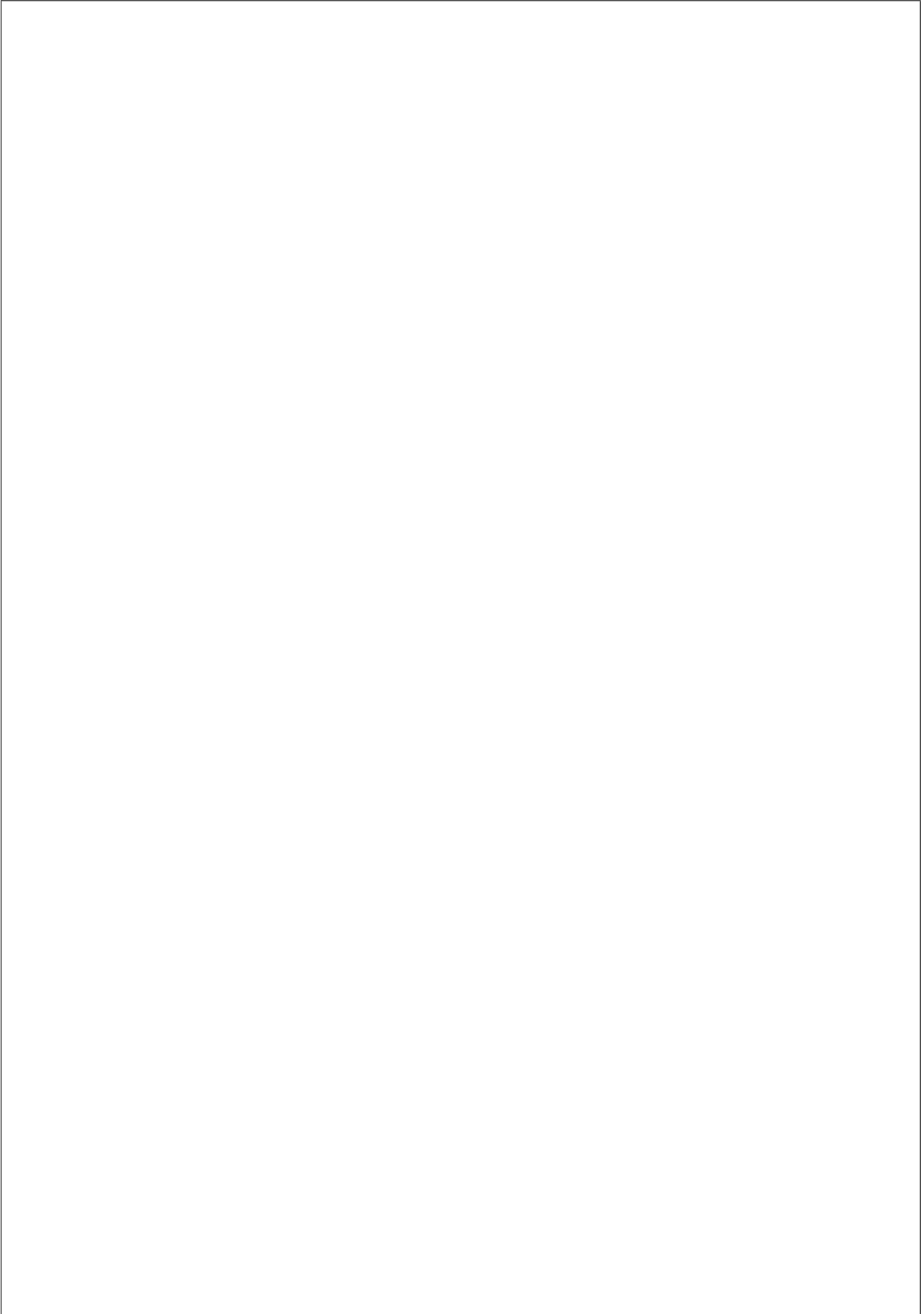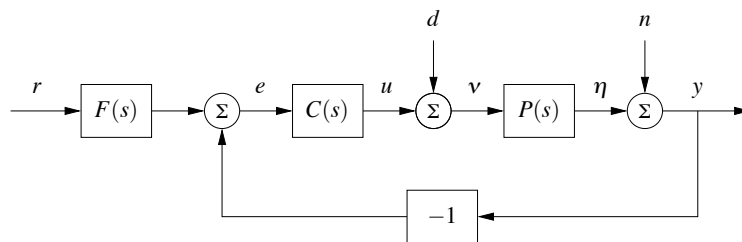
Figure 2: Negative feedback block diagram considered in Problem 4, with $F(s) = 1$.

4. [15 pts] Consider again the normalized turbine-governor dynamics of a synchronous generator given by[3]

$$
\begin{aligned}
\dot{x} &= Ax + Bu = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \nu \\
\eta &= Cx = \begin{bmatrix} 1 & 0 \end{bmatrix} x.
\end{aligned}
$$

(a) [2pts] Determine the transfer function $P(s)$ of the system.

(b) [2pts] Consider the negative feedback control system in Figure 2, with

$$
F(s) = 1, \quad n = 0, \quad d = 0.
$$

Let $P(s)$ be as in the previous question and $C(s)$ be the P controller

$$
C(s) = k_p.
$$

Determine all the values of the gain $k_p$ such that the closed-loop system is asymptotically stable.

**Hint** If you could not find any transfer function in (b) let $P(s) = \frac{1}{(s+2)^2}$.

(c) [2pts] Determine the transfer function $G_{yr}(s)$ from the reference $r$ to the output $y$ and the steady state output response to a step reference input when $n = d = 0$.

(d) [2pts] For this question and the following one, let $k_p$ be a parameter (hence, if you did not determine the values of $k_p$ in the previous question, you can still answer these questions). Determine the transfer function $G_{yn}(s)$ from the noise $n$ to the output $y$ and the steady state output response to a step noise $n$ when $r = d = 0$.

(e) [2pts] Determine the transfer function $G_{yd}(s)$ from the load disturbance $d$ to the output $y$ and the steady state output response to a step load disturbance $d$ when $r = n = 0$. In this case the load disturbance models a non-zero frequency deviation.

(f) [3pts] Consider now a PID controller
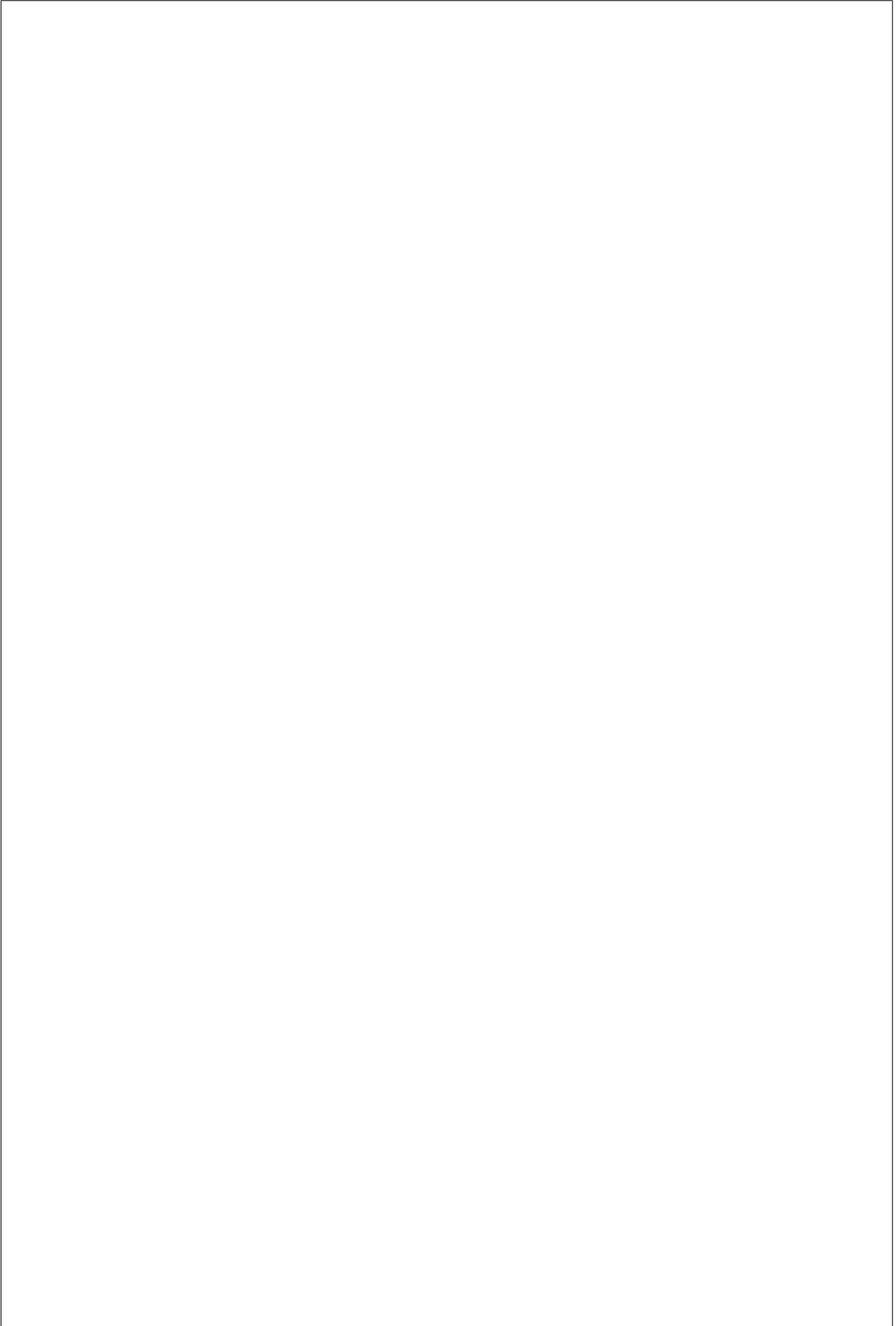
$$
C(s) = \frac{k_i + k_p s + k_d s^2}{s}.
$$

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[3]Notice that now the input is renamed as $\nu$ and the output as $\eta$ to be consistent with Figure 2.
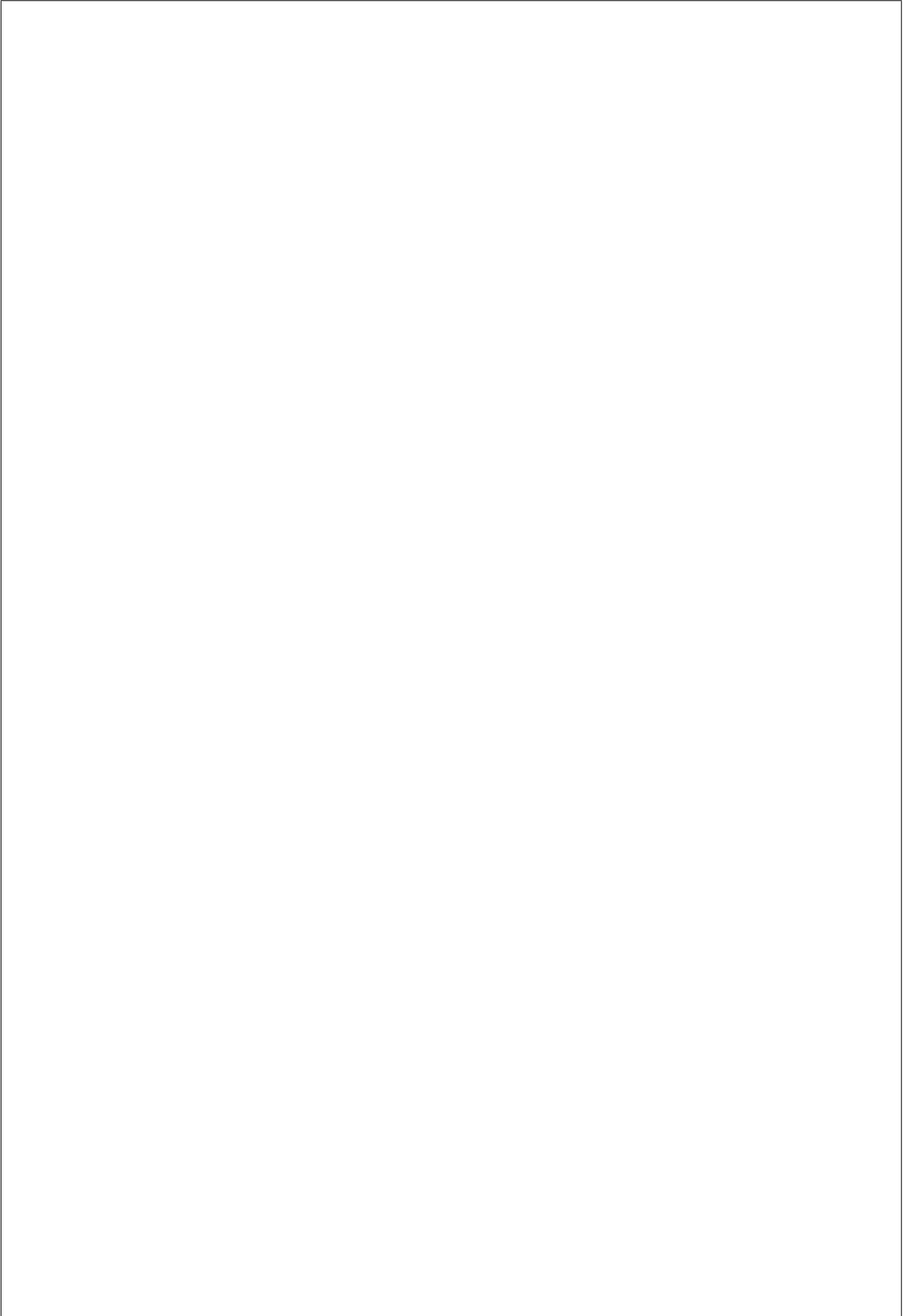
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Determine the parameters $k_p, k_d, k_i$ in such a way that the closed-loop system transfer function $G_{yr}(s)$ (with $d = n = 0$) has the denominator equal to

$$(s^2 + 2\zeta\omega_0 s + \omega_0^2)(s + a)$$

where $a = 5\zeta\omega_0$.

(g) [2pts] Determine the steady state output response to a step load disturbance $d$ when the controller is a PID and when $r = n = 0$. Is the controller able to guarantee the converge of $y$ to $r$ in spite of a constant frequency deviation $d$?

5. [10pts] Consider a negative feedback system as in Figure 2, where the process transfer function is
$$P(s) = \frac{1}{(s+1)^2}.$$

(a) [2pts] Design a controller $C(s)$ with as little number of poles as possible such that the closed-loop system has

   i. a zero steady state error response to a step input and

   ii. a constant $e_{steady}$ steady state error response to a ramp input such that $|e_{steady}| \leq 1$.

(b) [2pts] Suppose that the resulting open-loop transfer function $L(s) = C(s)P(s)$ has the Bode diagrams represented in Figure 3. Using the detailed (magnified) Bode diagrams in Figure 4, determine the gain crossover frequency $\omega_{gc}$ and the phase $\angle L(i\omega_{gc})$.

(c) [2pts] Using the Bode diagrams represented in Figure 3, qualitatively draw the corresponding Nyquist plot. Determine whether the system is asymptotically stable giving $P, N, Z$ ($P$ the number of poles of $P(s)$ with positive real parts, $N$ the net number of clockwise encirclements of $-1$, $Z$ the number of poles of the closed-loop system with positive real parts). Explain.

(d) [2pts] Determine the phase margin.

(e) [2pts] Determine the gain margin.

(f) [4pts] **(Bonus)** Suppose that a stricter specification is given, namely that the constant steady state error response $e_{steady}$ to a ramp input should satisfy $|e_{steady}| \leq 0.1$. How would you modify the previous controller to have the new specification satisfied? Using the Bode diagram in Figure 3, specify what would be the new gain cross-over frequency and the new (approximate) phase margin and specify whether or not the system would be asymptotically stable. If not, which controller would you use to stabilize the system while fulfilling the new specification on the steady state?

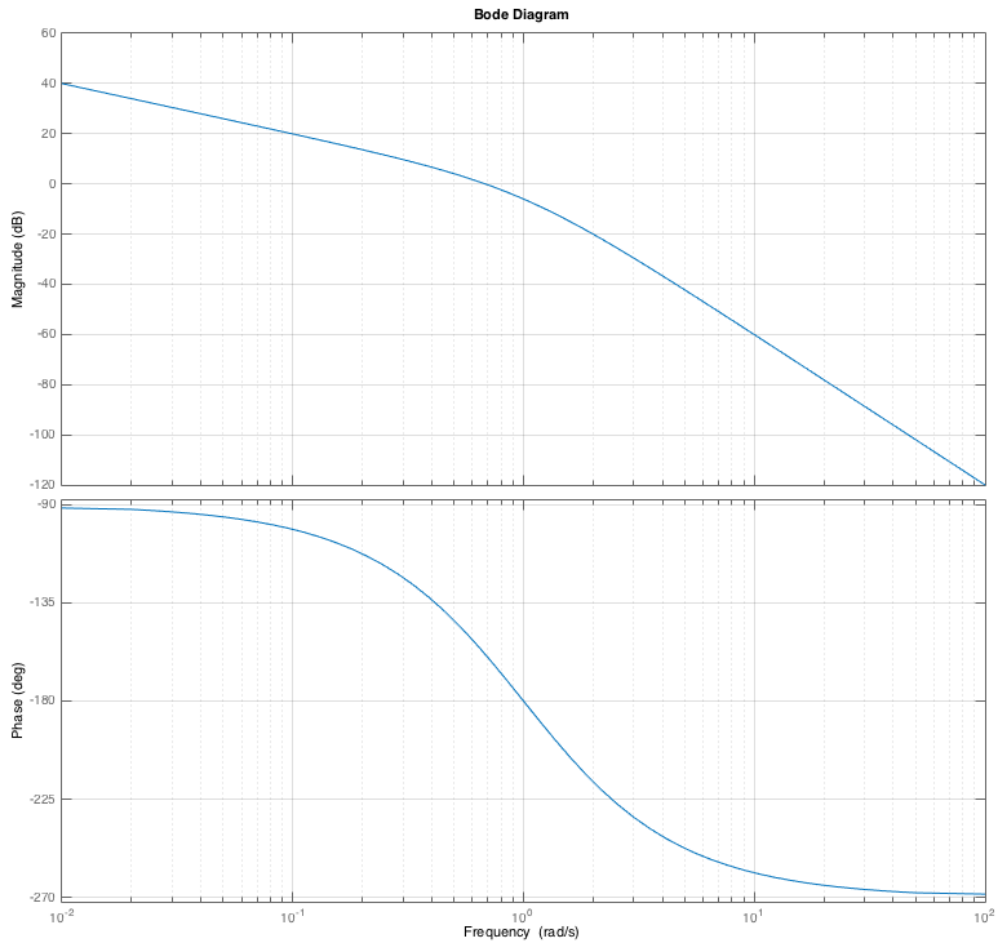PLEASE GO TO PAGE 19 FOR ADDITIONAL SPACE

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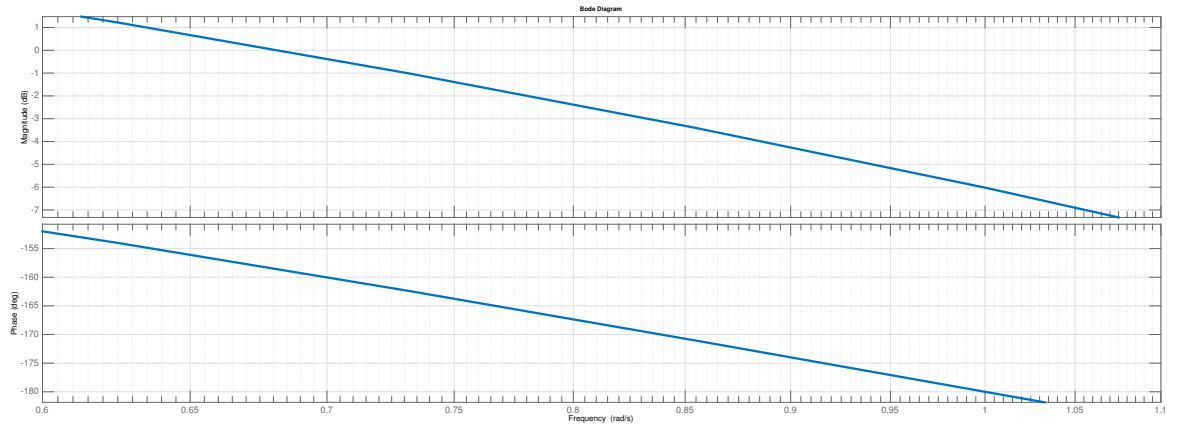Figure 3: Bode diagram of the closed-loop transfer function $L(s) = C(s)P(s)$.

Figure 4: Magnified picture of the Bode diagram of the loop transfer function $L(s) = C(s)P(s)$.